Inverse and Composite Rules

In 2019-20 I have not gone through the following derivation of Theorems 3.1.12 and 3.1.13 hence I would not expect you to know the proofs. I would, though, expect you to know, and be able to use, the two results.

For a further two rules consider the situation

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}.$$

We will write g as a function of $y \in \mathbb{R}$ and f a function of $x \in \mathbb{R}$.

Assume that g is defined on a neighbourhood of $k \in \mathbb{R}$, and f is defined on a neighbourhood of $\ell = g(k)$. In particular $f(g(k))$ is defined. Thus

$$k \overset{g}{\longmapsto} g(k) = \ell \overset{f}{\longmapsto} f(\ell) = f(g(k))\,.$$

Assume that f is differentiable at ℓ. For x in this neighbourhood of ℓ define the new function

$$F_\ell(x) = \begin{cases} \dfrac{f(x) - f(\ell)}{x - \ell} & \text{if } x \neq \ell, \\[2ex] \dfrac{df}{dx}(\ell) & \text{if } x = \ell. \end{cases}$$

Then

$$\lim_{x \to \ell} F_\ell(x) = \lim_{x \to \ell} \frac{f(x) - f(\ell)}{x - \ell} = \frac{df}{dx}(\ell) = F_\ell(\ell)\,, \tag{1}$$

and so F_ℓ is continuous at $x = \ell$.

We can rearrange the first line in the definition of $F_\ell(x)$ as

$$f(x) - f(\ell) = F_\ell(x)\,(x - \ell) \tag{2}$$

for $x \neq \ell$. But when $x = \ell$ both sides are zero, i.e. equal, Thus (2) holds for **all** x in the neighbourhood of ℓ. Use (2) for those x in the image of g, i.e. $x = g(y)$ for some y in a neighbourhood of k. Then

$$f(g(y)) - f(g(k)) = F_\ell(g(y))\,(g(y) - g(k))\,,$$

for such y. Our required formula is

$$\frac{f(g(y)) - f(g(k))}{y - k} = F_\ell(g(y)) \left(\frac{g(y) - g(k)}{y - k} \right), \tag{3}$$

for y in some **deleted** neighbourhood of k.

We make two important applications of (3) ; the first is that the inverse of a differentiable function is differentiable and the second is that the composition of two differentiable functions is differentiable. You should know both results from School but only now have you a justification of them.

Theorem 3.1.12 ***Inverse Rule*** *Suppose that $f(x)$ is strictly monotonic and continuous on a closed and bounded interval $[a,b]$. Write*

$$[c,d] = \begin{cases} [f(a)\,,\, f(b)] & \text{if } f \text{ is increasing} \\ [f(b)\,,\, f(a)] & \text{if } f \text{ is decreasing.} \end{cases}$$

By the Inverse Function Theorem there exists a strictly monotonic, continuous function $g : [c,d] \to [a,b]$ which is the inverse function of f so, if $y = f(x)$ then $x = g(y)$.

Suppose that f is differentiable at $\ell \in (a,b)$ with $df/dx \neq 0$ at $x = \ell$. Write $k = f(\ell)$ so $\ell = g(k)$ and $k \in (c,d)$.

Then g is differentiable at k and

$$\left.\frac{dg(y)}{dy}\right|_{y=k} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=\ell}}, \quad i.e. \quad \frac{dg}{dy}(k) = \frac{1}{\frac{df}{dx}(\ell)} = \frac{1}{\frac{df}{dx}(g(k))}.$$

If f is differentiable on (a,b) with $df/dx \neq 0$ at all points of (a,b), then g is differentiable on (c,d) and

$$\frac{dg(y)}{dy} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=g(y)}} = \frac{1}{\frac{df(g(y))}{dx}},$$

for all $y \in (c,d)$.

Proof *Not given in 2019-20.* Because f and g are inverses we have

$$\frac{f(g(y)) - f(g(k))}{y-k} = \frac{y-k}{y-k} = 1$$

for $y \in [c,d]\,, y \neq k$ and so (3) becomes

$$F_\ell\,(g(y)) \left(\frac{g(y) - g(k)}{y-k} \right) = 1, \tag{4}$$

for such y.

We now wish to divide by $F_\ell\,(g(y))$ but we can only do so if it is *non-zero*. Yet, as the inverse of a continuous function, $g(y)$ is continuous at $y = k$. We have noted above that $F_\ell\,(x)$ is continuous at $x = \ell = g(k)$. Hence, by the Composite Rule for continuous functions, $F_\ell\,(g(y))$ is continuous at $y = k$.

Therefore, the limit of this function at k equals the value of the function at k, i.e.

$$\lim_{y\to k} F_\ell\left(g(y)\right) = F_\ell\left(g(k)\right) = F_\ell\left(\ell\right) = \frac{df}{dx}(\ell)\,, \tag{5}$$

by (1).

By Lemma 3.1.8 if the value of a continuous function is non-zero at a point k then the function is non-zero in some *neighbourhood of* k. An assumption of the present theorem is that $df/dx \neq 0$ at $x = \ell$, i.e.

$$\lim_{y\to k} F_\ell\left(g(y)\right) \neq 0.$$

Thus there exists $\delta > 0$ such that if $k - \delta < y < k + \delta$ then $F_\ell\left(g(y)\right) \neq 0$. For such y we get, from (4),

$$\frac{g(y) - g(k)}{y - k} = \frac{1}{F_\ell\left(g(y)\right)}.$$

Then use the Quotient Rule for limits, along with (5) to deduce

$$\lim_{y\to k} \frac{g(y) - g(k)}{y - k} = \frac{1}{\lim_{y\to k} F_\ell\left(g(y)\right)} = \frac{1}{\dfrac{df}{dx}(\ell)}.$$

Since the limits exists g is differentiable at $x = k$ with derivative shown. ■

Example 3.1.13 *of* ***Inverse Rule****. Prove that*

$$\frac{d}{dy}\ln y = \frac{1}{y}$$

for all $y > 0$.

Solution Here $g(y) = \ln y$, which has been defined as the inverse of the strictly monotonic, continuous function $f(x) = e^x$. We know that $f'(x) = e^x$ so

$$\frac{d}{dy}\ln y = \frac{dg(y)}{dy} = \frac{1}{\left.\dfrac{df}{dx}(x)\right|_{x=g(y)}} = \frac{1}{\left.e^x\right|_{x=\ln y}} = \frac{1}{y},$$

as required. ■

Our second application of (3) is within

Theorem 3.1.14 ***Chain or Composite Rule*** *If $g(y)$ is differentiable at $y = k$ and $f(x)$ is differentiable at $x = g(k)$ then $(f \circ g)(y)$ is differentiable at $y = k$ and*

$$\frac{d(f \circ g)}{dy}(k) = \frac{df}{dx}(g(k))\frac{dg}{dy}(k).$$

Proof *Not given in* 2019-20 Let $y \to k$ in

$$\frac{f(g(y)) - f(g(k))}{y-k} = F_\ell(g(y))\left(\frac{g(y)-g(k)}{y-k}\right),$$

where $\ell = g(k)$. We need to use the Product Rule for limits on the right hand side which requires knowing that both

$$\lim_{y\to k} F_\ell(g(y)) \quad \text{and} \quad \lim_{y\to k}\frac{g(y)-g(k)}{y-k}$$

exist. The second limit exists since we are told $g(y)$ is differentiable at $y = k$.

To show the first limit exists we use the Composition Rule for continuous functions. This requires $g(y)$ to be continuous at $y = k$, which follows since g is differentiable there, and $F_\ell(x)$ to be continuous at $x = g(k)$, which followed from (1). The Composite Rule then gives

$$\lim_{y\to k} F_\ell(g(y)) = F_\ell\left(\lim_{y\to k} g(y)\right) = F_\ell(\ell) = \frac{df}{dx}(\ell).$$

Hence the Product Rule (allowable since all limits exist) can be used to give

$$\begin{aligned}\lim_{y\to k}\frac{f(g(y)) - f(g(k))}{y-k} &= \lim_{y\to k} F_\ell(g(y)) \lim_{y\to k}\left(\frac{g(y)-g(k)}{y-k}\right)\\ &= \frac{df}{dx}(\ell)\frac{dg}{dy}(k) \quad \text{by (1)}\\ &= \frac{df}{dx}(g(k))\frac{dg}{dy}(k) \quad \text{since } \ell = g(k).\end{aligned}$$

■

Note there is a common mistake made by far too many students attempting to prove the Chain Rule. See the Appendix for details.

Though we have been careful in the proof and statement to consider f as a function of x and g a function on y, in use we think of g and f as both functions of x and write

$$\frac{d\,(f \circ g)}{dx}(x) = \frac{df}{dx}(g(x))\,\frac{dg}{dx}(x)\,, \quad \text{i.e.} \quad (f \circ g)'(x) = f'(g(x))\,g'(x)\,,$$

for all x for which $f'(g(x))$ and $g'(x)$ exist.

Example 3.1.15 *of **Composite Rule**. For $y > 0$ and $\alpha \in \mathbb{R}$ we defined $y^\alpha = e^{\alpha \ln y}$. Prove, using this definition, that*

$$\frac{d}{dy} y^\alpha = \alpha y^{\alpha - 1},$$

for all $y > 0$.

Solution In the notation of the Chain Rule Theorem we have $f(x) = e^x = \exp(x)$ and $g(y) = \alpha \ln y$, so

$$f(g(y)) = \exp(g(y)) = \exp(\alpha \ln y) = y^\alpha.$$

We know that

$$\frac{df}{dx}(x) = e^x = \exp(x) \quad \text{and} \quad \frac{dg}{dy}(y) = \frac{\alpha}{y}$$

by Lemmas 3.1.4 and 3.1.13 respectively. Hence

$$\frac{d}{dy} y^\alpha = \frac{d\,(f \circ g)}{dy}(y) = \frac{df}{dx}(g(y))\,\frac{dg}{dy}(y) = \exp(g(y))\,\frac{\alpha}{y} = y^\alpha \frac{\alpha}{y} = \alpha y^{\alpha - 1}$$

as required. ■

Though we have been careful in the proof and statement to consider f as a function of x and g a function on y, in applications we think of g and f as **both** functions of x and write

$$\frac{d\,(f \circ g)}{dx}(x) = \frac{df}{dx}(g(x))\,\frac{dg}{dx}(x)\,, \quad \text{i.e.} \quad (f \circ g)'(x) = f'(g(x))\,g'(x)\,,$$

for all x for which $f'(g(x))$ and $g'(x)$ exist.